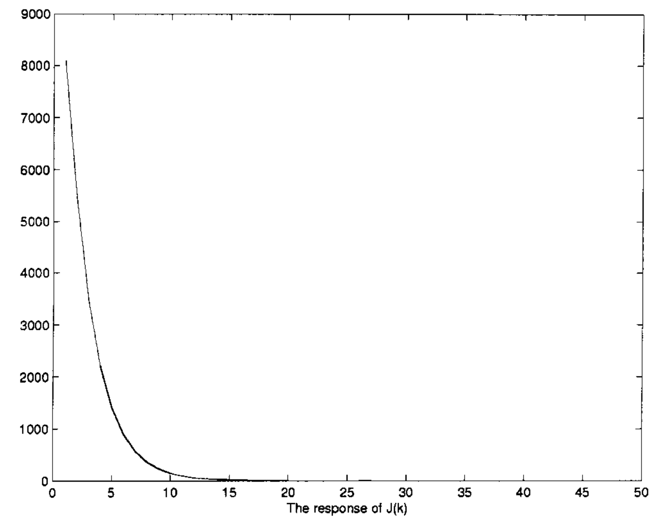

Fig. 4. The response of the performance function $J(u_k)$.

function has almost only one peak. This means that the decreasing direction for the entropy is toward having a narrow shape of the output probability density function. This is in line with the theoretical analysis as a small entropy means the uncertainty of the output is small, which indicates that the shape of the output probability density function is narrow. The response of the performance function is shown in Fig. 4, where it can be seen that a monotonically decreasing of $J(u_k)$ has been achieved. This shows the effectiveness of the proposed control algorithm (22).

VI. Conclusion

In this note, the entropy concept has been applied to the design of controllers for general stochastic systems subjected to arbitrary bounded random inputs. A linear B-spline model [11] has been used to formulate the system and a performance function which includes the entropy term and a quadratic input constraint. A local optimal control input has been formulated under certain conditions, and it has been shown that the control input is in a form of a nonlinear feedback which is related to the measured output probability density functions of the system and the past inputs. The closed loop stability is analyzed and a local stability condition has been established. A simulated example is used to illustrate the use of the proposed algorithm and encouraging results have been obtained.

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Mesh Stability of Look-Ahead Interconnected Systems

Aniruddha Pant, Pete Seiler, and Karl Hedrick

***Abstract*—In this note, we define a notion of mesh stability for a class of interconnected nonlinear systems. Intuitively mesh stability is the property of damping disturbance propagation. We derive a set of sufficient conditions to assure mesh stability of "look-ahead" interconnected systems. Mesh stability is shown to be robust with respect to structural and singular perturbations. The theory is applied to an example of formation flying.**

***Index Terms*—Interconnected systems, Lyapunov stability, string stability.**

I. Introduction

Intuitively mesh stability is the property of damping disturbances as they travel away from the source in an interconnected system. A significant amount of research has been done on the concept of string stability, a one dimensional counterpart of mesh stability. For a detailed literature review, we refer the reader to Swaroop [8]. Recently, while studying the position control of mobile offshore bases Hedrick considered strings which can move in multiple dimensions [11]. Seiler [12] introduced the concept of mesh stability and analyzed it in the linear case. The motivation for the problem comes from the analysis and the design of controllers for formation flying of unmanned aerial vehicles. In a formation one wants controllers to be designed so that any shock-wave arising from disturbance propagation should dampen as it travels away from the source. In other words we want the closed loop interconnected system for the formation mesh stable.

Swaroop considered sufficient conditions similar to this note. They assumed infinite strings of dynamical systems. This work (a) generalizes the interconnection structure to finite dynamical systems, (b) gives tighter bounds on the interconnections, and (c) develops a general methodology for analyzing mesh stability. We obtain sufficient condi-

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tions for exponential mesh stability of a class of interconnected nonlinear systems.

II. Preliminaries

A look ahead system: An interconnected system, is called look-ahead, if the (i,j)th subsystem is connected only to the subsystems (k,l) such that $k \le i$ and $l \le j$. Consider a system of ordinary differential equations

$$\dot{x}_{i,j} = f_{i,j}(x_{i,j},\ldots,x_{1,j},x_{i,j-1},\ldots,x_{i,1},\ldots,x_{1,1}). \tag{1}$$

For the convenience of notation, define a mapping as shown in

$$m(i,j) = \begin{cases} j + \sum_{p=1}^{L-1} p & \text{for } (i+j-1) \le N \\ (N-i+1) + \frac{(N)(N+1)}{2} + \sum_{p=N+1}^{L-1}(2N-p) & \\ \text{else} & \end{cases}. \tag{2}$$

If we write the dynamics of the subsystems with the new index m and rename the index, we will get a system which is of the form

$$\dot{x}_i = f_i(x_i, x_{i-1},\ldots,x_1) \tag{3}$$

where $i \in \{1,\ldots,N\}, x_i \in \mathbb{R}^n, f : \mathbb{R}^n \times \cdots \times \mathbb{R}^n \to \mathbb{R}^n$ and $f(0,\ldots,0)=0$. If $x_i(t) \in \mathbb{R}^n$ is vector function of time, then $\|x_i(t)\|$ denotes any vector norm of $x_i(t)$ and $x = [x_1^T \ldots x_n^T]^T \in \mathbb{R}^{N\cdot n}$. Other notation we use is as follows:

$$\|x(0)\|_\infty = \max_i \|x_i(0)\|$$
$$\|x(t)\|_\infty = \max_i \|x_i(t)\|_\infty$$
$$\|x(t)\|_\infty^k = \max_{i\le k} \|x_i(t)\|_\infty.$$

Exponential Mesh Stability: The origin $x = 0$ of the dynamical system (3), is globally exponentially mesh stable if (a) given any $\epsilon > 0, \exists\, \delta > 0$ such that, $\|x(0)\|_\infty < \delta \implies \|x(t)\|_\infty < \epsilon$ (b) $x \to 0$, exponentially $\forall x \in \mathbb{R}^n$, and (c) $\|x_i(t)\|_\infty \le \|x(t)\|_\infty^{i-1}, \forall i \in \{2,\ldots,N\}$.

Converse Lyapunov Theorem: [9] If $\dot{x} = f(x), x \in \mathbb{R}^n$ is globally exponentially stable, $f(x)$ is continuously differentiable, and $\partial f/\partial x$ is bounded over $\mathbb{R}^n$, then there exists a C^1 function $V(x) : \mathbb{R}^n \to \mathbb{R}_+$ and constants $\alpha_h, \alpha_l, \alpha_1, \alpha_2 > 0$ such that $\forall x \in \mathbb{R}^n$

$$\alpha_l\|x\|^2 \le V(x) \le \alpha_h\|x\|^2$$
$$\frac{\partial V}{\partial x} f(x) \le -\alpha_1\|x\|^2$$
$$\left\|\frac{\partial V}{\partial x}\right\| \le \alpha_2\|x\|.$$

Comparison Principle: A special form of comparison principle for vector Lyapunov functions is presented. For a general statement, see [5]. Let $r(t) \in \mathbb{R}^m$ be the unique solution of the vector differential equation, $\dot{r} = Ar$ with $r(0) = r_0$. $A \in \mathbb{R}^{m\times m}$ and $a_{i,j} \ge 0$, when $i \ne j$. If we have a C^1 function $v(t) \in \mathbb{R}^m$ that satisfies $\dot{v} \le Av$ then

$$v(0) \le r_0 \implies v(t) \le r(t), \quad \forall t \in [0,\infty).$$

III. Mesh Stability of Look Ahead Systems

Theorem 1: Consider the interconnected system (3), satisfying the following.

- f is globally Lipschitz in its arguments, i.e.,

$$\|f_i(y_i,\ldots,y_1) - f_i(x_i,\ldots,x_1)\| \le l_1\|y_1 - x_1\| + \cdots + l_i\|y_i - x_i\|, \quad l_i \in \mathbb{R}^+.$$

- The origin of the *free* ith subsystem $\dot{x}_i = f_i(x_i,0,\ldots,0)$ is globally exponentially stable.

Then, for sufficiently small $l_i > 0$, the interconnected system is globally exponentially mesh stable.

Proof: The converse Lyapunov theorem implies the existence of a Lyapunov function $V_i(x_i)$ for the free system and

$$\begin{aligned}\dot{V}_i &= \frac{\partial V_i}{\partial x_i} f_i(x_i,0,\ldots,0) \\ &\quad + \frac{\partial V_i}{\partial x_i}[f_i(x_i,\ldots,x_1) - f_i(x_i,0,\ldots,0)] \\ &\le -\alpha_1\|x_i\|^2 + \alpha_2\|x_i\|\left(\sum_{j=1}^{i-1} l_j\|x_j\|\right).\end{aligned} \tag{4}$$

Using the inequality $xy \le (x^2+y^2)/2$ and the converse Lyapunov theorem

$$\dot{V}_i \le -\alpha_1\|x_i\|^2 + \frac{\alpha_2}{2}\left(\sum_{j=1}^{i-1} l_j(\|x_i\|^2 + \|x_j\|^2)\right)$$
$$\dot{V}_i \le -\frac{\left(\alpha_1 - \frac{\alpha_2}{2}\sum_{j=1}^{i-1} l_j\right)}{\alpha_h} V_i + \frac{\alpha_2}{2\alpha_l}\sum_{j=1}^{i-1} l_j V_j. \tag{5}$$

Define a matrix $B \in \mathbb{R}^{N\times N}$ and $V := [V_1\ V_2 \cdots V_N]^T : \mathbb{R}^{N\cdot n} \to \mathbb{R}_+^N$ such that

$$b_{i,i} = -\frac{\left(\alpha_1 - \frac{\alpha_2}{2}\sum_{j=1}^{i-1} l_j\right)}{\alpha_h}, \quad b_{i,j} = \frac{\alpha_2}{2\alpha_l} l_j, \quad i > j.$$

Then inequality (5) can be written as

$$\dot{V}(t) \le BV(t). \tag{6}$$

One can see that if $\alpha_1 - (\alpha_2/2)\sum_{j=1}^{i-1} l_j > 0$ then B is a stable triangular matrix. For $V(0) \le z(0)$ where $\dot{z} = Bz$ and using the comparison theorem

$$0 \le \lim_{t\to\infty} V(t) \le \lim_{t\to\infty} z(t) = 0$$

This proves exponential stability for (3). From (4), we have

$$\dot{V}_i \le -\|x_i\|\left(\alpha_1\|x_i\| - \alpha_2\left(\sum_{j=1}^{i-1} l_j\right)\|x\|_\infty^{i-1}\right). \tag{7}$$

Let

$$g_k := \frac{\alpha_2}{\alpha_1}\left(\sum_{j=1}^{k-1} l_j\right)\|x\|_\infty^{k-1}, \quad c := \left(\frac{\alpha_h}{\alpha_l}\right)^{1/2} \ge 1. \tag{8}$$

Define the following sets in state space:

$$B_i := \{x_i : \|x_i\| \le g_i\}, \quad B_{ir} := \{x_i : \|x_i\| \le c\cdot g_i\}.$$

If $x_i \in \mathbb{R}^n \backslash B_i$, then $\dot{V}_i$ is negative. Thus, for all $x \in \mathbb{R}^n\backslash B_{ir}$ and $\forall y \in B_i$, we have

$$V(y) \le \alpha_h\|y\|^2 < \frac{1}{c^2}\cdot\alpha_h\|x\|^2 = \alpha_l\|x\|^2 \le V(x).$$

This implies

$$\begin{aligned} x_i(0) \in B_i &\implies x_i(t) \in B_{ir}, \quad \forall t \\ &\implies \|x_i\| \le \left(\frac{\alpha_h}{\alpha_l}\right)^{1/2}\frac{\alpha_2}{\alpha_1}\left(\sum_{j=1}^{i-1} l_j\right)\|x\|_\infty^{i-1}, \quad \forall t \\ &\implies \|x_i\|_\infty \le \left(\left(\frac{\alpha_h}{\alpha_l}\right)^{1/2}\left(\frac{\alpha_2}{\alpha_1}\right)\sum_{j=1}^{i-1} l_j\right)\|x\|_\infty^{i-1}. \end{aligned} \tag{9}$$

Defining $k = [((\alpha_h)/(\alpha_l))^{1/2}((\alpha_2)/(\alpha_1))\sum_{j=1}^{i-1} l_j]$, one can see that for small enough $\sum_{j=1}^{i-1} l_j, k < 1$. This proves (3) is globally exponentially mesh stable. □

A. Robustness to Structural Perturbations

Consider

$$\dot{x}_i = f_i(x_i, x_{i-1}, \ldots, x_1) + \epsilon f_i^p(x_i, x_{i-1}, \ldots, x_1) \tag{10}$$

and assume $f_i^p(0, \ldots, 0) = 0$. Following the procedure similar to the previous section:

$$\dot{V}_i \leq -(\alpha_1 - \epsilon \alpha_2 l_1^p) \|x_i\|^2 + \|x_i\| \sum_{j=1}^{i-1} \alpha_2 \left(l_j + \epsilon l_j^p\right) \|x_j\|.$$

Here, l_i^p are the Lipchitz constants for the perturbation. Notice the similarity between the above equation and (4). If $\epsilon, l_j^p, \forall i = 1, \ldots, N$ are small enough, then we have mesh stability of the perturbed system, provided that the unperturbed system is mesh stable. This analysis shows us that if the feedback linearized system is mesh stable then as long as modeling error is small enough, we have mesh stability of the original nonlinear system.

B. Robustness to Singular Perturbations

In this section, we demonstrate that if the size of a singular perturbation is small, then mesh stability is preserved. First, we state a classical theorem regarding exponential stability of perturbed nonlinear systems [6].

Theorem 2: Consider an autonomous system

$$\dot{x} = f(x, z) \quad \epsilon \dot{z} = g(x, z) \tag{11}$$

where $x \in \mathbb{R}^n, z \in \mathbb{R}^m$. Assume that the origin is the equilibrium point and the functions f and g are globally Lipschitz in their arguments. Let $z = h(x)$ be the solution of $g(x, z) = 0$, such that $h(0) = 0$. Let $y := z - h(x)$. $\epsilon \dot{z} = g(x, z)$ is called the *boundary layer system* and $\dot{x} = f(x, h(x))$ is called the *reduced system*.

Suppose the following conditions are satisfied.

- The reduced system is globally exponentially stable i.e., there exist, constants $\alpha_l, \alpha_h, \alpha_1, \alpha_2 > 0$ and a function $V(x)$ s.t. $\forall x \in \mathbb{R}^n$

$$\alpha_l \|x\|^2 \leq V(x) \leq \alpha_h \|x\|^2$$
$$\frac{\partial V}{\partial x} f(x, h(x)) \leq -\alpha_1 \|x\|^2 \left\| \frac{\partial V}{\partial x} \right\| \leq \alpha_2 \|x\|.$$

- The boundary layer system is globally exponentially stable, uniformly for each frozen x. This implies that there exist constants $\beta_l, \beta_h, \beta_1, \beta_2, \beta_3, \beta_4 > 0$ and a function $W(x, y)$ such that $\forall x \in \mathbb{R}^n, \forall y \in \mathbb{R}^m$,

$$\beta_l \|y\|^2 \leq W(x, y) \leq \beta_h \|y\|^2$$
$$\frac{\partial W}{\partial y} g(x, y + h(x)) \leq -\beta_1 \|y\|^2$$
$$\left\| \frac{\partial W}{\partial x} \right\| \leq \beta_2 \|y\|, \quad \left\| \frac{\partial W}{\partial y} \right\| \leq \beta_3 \|y\|, \quad \left\| \frac{\partial h}{\partial x} \right\| \leq \beta_4.$$

- The above conditions also imply that there exist constants $\gamma_1, \gamma_2 > 0$ such that

$$\left[\frac{\partial W}{\partial x} - \frac{\partial W}{\partial y} \frac{\partial h}{\partial x}\right] f(x, y + h(x)) \leq \gamma_1 \|x\| \|y\| + \gamma_2 \|y\|^2. \tag{12}$$

Then there exists $\epsilon^* > 0$ s.t. the perturbed system is exponentially stable for $\epsilon < \epsilon^*$. For a proof, see [6].

Consider the ith *free, perturbed* system

$$\dot{x}_i = f_i(x_i, z_i), \quad \epsilon \dot{z}_i = g_i(x_i, z_i)$$

Assuming the reduced and the boundary layer systems are exponentially stable, by the above theorem, for small enough ϵ, the perturbed system is exponentially stable. Consider the interconnected subsystems $i \in \{1, \ldots, N\}$,

$$\dot{x}_i = f_i(x_i, z_i, x_{i-1}, \ldots, x_1), \quad \epsilon \dot{z}_i = g_i(x_i, z_i). \tag{13}$$

The reduced ith system is given by

$$\dot{x}_i = f_i(x_i, h_i(x_i), x_{i-1}, \ldots, x_1).$$

From the results in the previous sections, we can argue that if the interconnection Lipschitz constants are small enough then the reduced interconnected system is exponentially stable, provided the reduced, free subsystems are exponentially stable to start with. Equation (13) can be rewritten in terms of concatenated vectors, $x := [x_1, \ldots, x_N] \in \mathbb{R}^{N \cdot n}$, and $z := [z_1, \ldots, z_N] \in \mathbb{R}^{N \cdot m}$.

$$\dot{x} = f(x, z) \quad \epsilon \dot{z} = g(x, z) \tag{14}$$

Applying the singular perturbation theorem, we can argue that the equilibrium of (14) is exponentially stable for small enough ϵ. Proceed to prove that mesh stability is robust to the singular perturbations. Observe that in the the effect of $z_j, j \neq i$, on ith subsystem of (13) is via x_j. Let V_i be a Lyapunov function for the reduced, free subsystem

$$\begin{aligned}
\dot{V}_i &= \frac{\partial V_i}{\partial x_i} f_i(x_i, z_i, x_{i-1}, \ldots, x_1) \\
&\leq -\alpha_1 \|x_i\|^2 + \alpha_2 \|x_i\| \left[l \|z_i - h_i(x_i)\| + \sum_{k=1}^{i-1} l_k \|x_k\| \right] \\
&= -\|x_i\| \left(\alpha_1 \|x_i\| - \alpha_2 \left[l \|z_i - h_i(x_i)\| + \sum_{k=1}^{i-1} l_k \|x_k\| \right] \right).
\end{aligned}$$

Therefore, using the analogous arguments as before

$$\|x_i\| \leq \left(\frac{\alpha_h}{\alpha_l}\right)^{1/2} \frac{\alpha_2}{\alpha_1} \left[l \|z_i - h_i(x_i)\| + \sum_{k=1}^{i-1} l_k \|x\|_\infty^{i-1} \right].$$

Define

$$\delta_\epsilon := \frac{l \|z_i - h_i(x_i)\|}{\|x\|_\infty^{i-1}}.$$

Observe that $\|x\|_\infty^{i-1} \geq \|x(0)\|_\infty^{i-1} > 0$

$$\implies \left(\frac{\alpha_h}{\alpha_l}\right)^{1/2} \frac{\alpha_2}{\alpha_1} \left[\sum_{k=1}^{i-1} l_k + \delta_\epsilon \right] < 1 \text{ for some } \epsilon > 0$$
$$\implies \text{Mesh Stability for (13).}$$

We show that $l \|z_i - h_i(x_i)\|$, and consequently, δ_ϵ can be made as small as we want by reducing ϵ. We have

$$\dot{y}_i = \frac{g_i(x_i, y_i + h_i(x_i))}{\epsilon} - \frac{\partial h_i}{\partial x_i} f_i(x_i, y_i + h_i(x_i), x_{i-1}, \ldots, x_1).$$

Applying the converse Lyapunov theorem and taking derivatives

$$\begin{aligned}
\dot{W} = &\frac{\partial W}{\partial y_i} \left(\frac{g_i(x_i, y_i + h_i(x_i))}{\epsilon} \right) \\
&- \frac{\partial W}{\partial y_i} \left(\frac{\partial h_i}{\partial x_i} f_i(x_i, y_i + h_i(x_i), x_{i-1}, \ldots, x_1) \right) \\
&+ \frac{\partial W}{\partial x_i} f_i(x_i, y_i + h_i(x_i), x_{i-1}, \ldots, x_1)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\partial W}{\partial y_i}\left(\frac{g_i(x_i, y_i + h_i(x_i))}{\epsilon}\right) \\
&\quad + \left(\frac{\partial W}{\partial x_i} - \frac{\partial W}{\partial y_i}\left(\frac{\partial h_i}{\partial x_i}\right)\right) f_i(x_i, y_i + h_i(x_i)) \\
&\quad + \left(\frac{\partial W}{\partial x_i} - \frac{\partial W}{\partial y_i}\left(\frac{\partial h_i}{\partial x_i}\right)\right) \\
&\quad \times (f_i(x_i, y_i + h_i(x_i), x_{i-1}, \ldots, x_1) \\
&\quad - f_i(x_i, y_i + h_i(x_i))).
\end{aligned}
$$

Applying the converse Lyapunov theorem and Lipschitz bounds

$$
\dot{W} \le -\frac{\beta_1}{\epsilon}\|y_i\|^2 + \gamma_1\|x_i\|\|y_i\| + \gamma_2\|y_i\|^2 + k\|y_i\|\sum_{j=1}^{i-1} l_j\|x_j\|.
$$

Using inequality $xy \le (x^2 + y^2)/2$, simplifying and renaming the constants we have

$$
\dot{W} \le -\frac{k_1}{\epsilon}\|y_i\|^2 + \sum_{k=1}^{i} \theta_k\|x_k\|^2.
$$

Using converse Lyapunov theorem bounds

$$
\dot{W} \le -\frac{k_1}{\beta_h \epsilon} W + \sum_{k=1}^{i} \frac{\theta_k}{\alpha_l} V_k
$$

We know from previous arguments that all the subsystems are exponentially stable, therefore there exist constants $M_k, b_k > 0$ such that

$$
V_k = M_k e^{-b_k t}, \quad \forall k \in [1, N].
$$

Substituting, simplifying, and renaming constants

$$
\begin{aligned}
\dot{W}_i(t) &\le -\frac{k_1}{\epsilon} W_i + \sum_{k=1}^{i} A_k e^{-b_k t} \\
&\Longrightarrow W_i \le e^{-\frac{k_1}{\epsilon} t} W_i(0) + \sum_{k=1}^{i} \frac{A_k}{\frac{k_1}{\epsilon} - b_k} e^{-b_k t}.
\end{aligned}
$$

For simplicity assume $W_i(0) = 0$.

$$
\Longrightarrow \|y_i\|^2 \le \sum_{k=1}^{i} \frac{A_k}{\beta_l\left(\frac{k_1}{\epsilon} - b_k\right)}.
$$

Let $A := \max_k A_k$ and $b := \max_k b_k$, then

$$
\|y_i\|^2 \le \frac{iA}{\beta_l\left(\frac{k_1}{\epsilon} - b\right)}, \quad \forall t \Longrightarrow \|y_i\|_\infty \le \left(\frac{iA}{\beta_l\left(\frac{k_1}{\epsilon} - b\right)}\right)^{1/2}.
$$

We can see that $\|y_i\|_\infty$, and correspondingly, δ_ϵ can be made as small as we want by reducing ϵ. Therefore, for small enough ϵ, the singularly perturbed system is mesh stable.

IV. EXAMPLE

In this section, we apply the theory developed in previous sections to an idealized example of formation flying in a plane. This example has been analyzed with the classical transfer function perspective in [12]. Consider a mesh of point masses shown in Fig. 1. Define the position, velocity and acceleration vectors as: $p_{i,j} = [y_{i,j}; z_{i,j}]^T$; $v_{i,j} = \dot{p}_{i,j}$; $a_{i,j} = \dot{v}_{i,j}$. Then the dynamics are given by, $\ddot{p}_{i,j} = a_{i,j}$ where

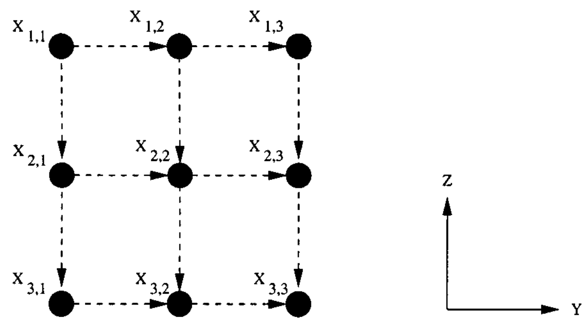

Fig. 1. Mesh Schematic.

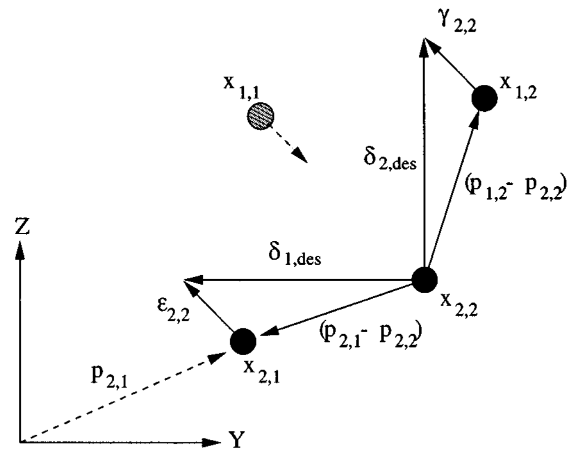

Fig. 2. Error Definitions.

$a_{i,j} \in \mathbb{R}^2$ is the input acceleration. Define the following mesh spacing errors:

$$
\epsilon_{i,j} := \delta_{1,\text{des}} - (p_{i,j-1} - p_{i,j}) \tag{15}
$$

$$
\gamma_{i,j} := \delta_{2,\text{des}} - (p_{i-1,j} - p_{i,j}) \tag{16}
$$

Fig. 2 shows the specific error vectors with $i = j = 2$. The $\epsilon_{i,j}$'s are the errors with respect to the 'left' neighbor along the row and the $\gamma_{i,j}$'s are the errors with respect to the 'above' neighbor along the column. $\delta_{1,\text{des}}$ and $\delta_{2,\text{des}}$ are the desired spacing vectors. To obtain the spatial arrangement shown in the schematic (Fig. 1), we define $\delta_{1,\text{des}} = [-L; 0]^T$ and $\delta_{2,\text{des}} = [0; L]^T$ for all i, j. Define a composite error vector, $e_{i,j} := \epsilon_{i,j} + \gamma_{i,j}$, at each point in the mesh. Define the following 2×1 vector sliding function:

$$
S_{i,j} = (e_{i,j}) + q_1 \cdot (\dot{e}_{i,j}) + q_2 \cdot (v_{i,j} - v_l) + q_3 \cdot (p_{i,j} - p_l). \tag{17}
$$

To normalize the control effort, the composite error for boundary subsystems is defined to be $e_{i,j} = 2\epsilon_{i,j}$ or $e_{i,j} = 2\gamma_{i,j}$, as appropriate. Choose the control input $a_{i,j}$ to force $\dot{S}_{i,j} := -KS_{i,j}$. This standard sliding surface control design makes $S_{i,j} \to 0$ exponentially and gives the following error dynamics for the mesh. For details, see [12]

$$
E_{i,j}(s) = \begin{bmatrix} H(s) & 0 \\ 0 & H(s) \end{bmatrix} \left[\frac{E_{i-1,j}(s) + E_{i,j-1}(s)}{2}\right]. \tag{18}
$$

With

$$
H(s) := \frac{\frac{1}{1+q_2}(s + q_1)}{s + \frac{q_1 + q_3}{1+q_2}}.
$$

Let $h(t)$ be the impulse response of $H(s)$. If $q_1q_2 > q_3$, $h(t)$ does not change sign [7], we have

$$\|h(t)\|_1 = \sup_\omega |H(j\omega)| = H(0) = \frac{q_1}{q_1 + q_3}.$$

$H(0) < 1$ if $q_3 > 0$, therefore, mesh stability follows [12]. This condition says that we need to have leader position information in our platoon control law to achieve mesh stability.

Next, we apply the theory developed in the previous section to this example and compare the results. Define

$$\alpha_1 := \frac{q_1 + q_3}{1 + q_2}, \quad \beta_1 := \frac{1}{1 + q_2}, \quad \alpha_2 := \frac{q_1}{1 + q_2}.$$

For simplicity consider a (2 × 2) mesh. Rename the indices as shown in (2), then the error dynamics for the first element of the 2 × 1 error vector of the $(2, 2)$ subsystem are given by

$$\dot{e}_4(1) = -\alpha_1 e_4(1) + \frac{(\alpha_2 - \alpha_1\beta_1)}{2} e_3(1) + \frac{(\alpha_2 - \alpha_1\beta_1)}{2} e_2(1) + \beta_1(\alpha_2 - \alpha_1\beta_1)e_1(1).$$

The error dynamics for the second term $e_4(2)$ in the error vector are given by the same equation. Assume $\alpha_2 - \alpha_1\beta_1 > 0, \beta_1 < 1$, which is equivalent to saying that $q_1q_2 > q_3$ and $q_2 > 0$ respectively. Applying the condition in (9). we have mesh stability if

$$(\alpha_2 - \alpha_1\beta_1)(1 + \beta_1) < \frac{(\alpha_2 - \alpha_1\beta_1)}{1 - \beta_1} < \alpha_1. \tag{19}$$

Substituting the values for $\alpha_1, \alpha_2, \beta_1$ in terms of q_1, q_2, q_3 yields mesh stability if $q_3 > 0$. Thus, the results agree with the results obtained by linear transfer function approach.

V. Conclusion

In this note, we gave a definition of mesh stability of finite, interconnected, look-ahead systems. Sufficient conditions for exponential mesh stability were presented by using Lyapunov theory of stability and the comparison theorem for vector Lyapunov functions. Mesh stability was shown to be robust with respect to structural and singular perturbations. An idealized example in planer motion illustrates the main result.

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An Improvement on "Delay and Its Time-Derivative Dependent Robust Stability of Time-Delayed Linear Systems With Uncertainty"

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***Abstract*—In this note, an improvement on the stability result in the above paper is given based on a modified Lyapunov function. It is shown that our result is much less conservative than that in the paper especially when the size of derivative of the time delay increases. For comparison, we give two examples.**

***Index Terms*—Uncertain system, delay, stability.**

In the above paper,[1] a stability criterion, which is dependent on the size of time-delay and the size of its derivative, was proposed for a class of uncertain systems with time varying delay. In this note, based on a modified Lyapunov function, we derive a new stability criterion for the system (1) in the paper.[1] For simplicity, all the symbols in this note are the same as the ones in the above paper[1]. Construct a Lyapunov function as

$$V(x,t) = v_0(x) + hv_1(x,t) + hv_2(x,t) \tag{1}$$

where v_0 and v_1 are given as in the paper.[1] Different from (12) in that paper.[1] we modify the $v_2(x,t)$ as

$$\begin{aligned} v_2(x,t) &= -\int_{t-\tau(t)}^{t}\int_{t-\tau(t)}^{s} x^T(\xi-\tau(\xi)) \\ &\quad \times \Omega_1(X_1,\eta_1)x(\xi-\tau(\xi))\,d\xi\,ds \\ &\quad + h\int_{t-\tau(t)}^{t} x^T(s-\tau(s))\Omega_1(X_1,\eta_1)x(s-\tau(s))\,ds \\ &\quad + \frac{h}{1-d}\int_{t-\tau(t)}^{t} x^T(s)\Omega_1(X_1,\eta_1)x(s)\,ds. \end{aligned} \tag{2}$$

Since $v_1(x,t)$ is positive–definite, if $v_2(x,t)$ is also positive definite, then, we can show that

$$V(x,t) \geq v_0(x) = x^T Px.$$

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